

# Renormalization of the Moment of Inertia for the Classical Electromagnetic One-body Problem

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*Dedicated to Professor PASCUAL JORDAN on the occasion of his 65th birthday*

A general process of renormalization of the energy-momentum vector and the angular-momentum tensor is given. It is shown that in the simple limiting case of a charged line segment rotating about a fixed point of symmetry, this process reduces to a renormalization of the moment of inertia analogous to the mass renormalization given by DIRAC for the limiting case of a point particle.

In a preceding paper<sup>1</sup> a general process of renormalization of the energy-momentum vector and the angular-momentum tensor is given. We show in this article that in the simple limiting case of a charged line segment rotating about a fixed point of symmetry, this process reduces to a renormalization of the moment of inertia analogous to the mass renormalization given by DIRAC for the limiting case of a point particle<sup>2</sup>. For completeness we include a derivation of DIRAC's result from the point of view of<sup>1</sup>.

In section I we briefly review the results of<sup>1</sup>. Since we are dealing with a non-conserved matter tensor, the choice of hyperplane over which we integrate to form the energy-momentum vector and angular-momentum tensor is important. In section II a canonical method of choosing a hyperplane at each point is given. Also a timelike curve to represent the body movement is defined. In section III we give some preliminary formulae and in sections IV and V we carry out the renormalization calculations in the two cases to be considered.

## I.

We consider an isolated extended charged body in a flat space-time and assume that it undergoes accelerated motion for only a finite interval of time. (See<sup>1</sup> for a discussion of this assumption.) Let  $\varrho$  be the charge density of the body. We assume the current to be of the form  $\varrho u^i$ .

Let  $\varphi_{\pm}^i$  be the advanced and retarded solutions of the wave equation

$$\square \varphi^i = 4 \pi \varrho u^i$$

which vanish at infinity. From these we form the advanced and retarded MAXWELL fields:

$$F_{\pm}^{ij} = \varphi_{\pm}^{i,i} - \varphi_{\pm}^{i,j}.$$

The MAXWELL stress-energy tensor is defined in terms of the retarded field:

$$-4 \pi \tau^{ij} = F_{-}^{ik} F_{-}^j{}_{-k} - \frac{1}{4} g^{ij} F_{-}^{kl} F_{-kl}. \quad (1)$$

Let  $T^{ij}$  be the matter tensor of the body. Using the identity

$$\tau^{ij}{}_{,j} = -F_{-}^{ij} \varrho u_j, \quad (2)$$

the conservation laws

$$(T^{ij} + \tau^{ij})_{,j} = 0 \quad (3)$$

may be written in the form

$$T^{ij}{}_{,j} = F_{-}^{ij} \varrho u_j. \quad (4)$$

Using the advanced as well as the retarded MAXWELL field, this equation may be rewritten as

$$T^{ij}{}_{,j} = \frac{1}{2} (F_{-}^{ij} + F_{+}^{ij}) \varrho u_j + \frac{1}{2} (F_{-}^{ij} - F_{+}^{ij}) \varrho u_j. \quad (5)$$

In<sup>1</sup> the tensor  $\tau^{ij}$ , defined by

$$-4 \pi \tau^{ij} = \frac{1}{2} (F_{-}^{ik} F_{+}^j{}_{+k} + F_{+}^{ik} F_{-}^j{}_{-k}) - \frac{1}{4} g^{ij} F_{-}^{kl} F_{+kl} \quad (6)$$

is introduced and it is shown that the divergence of  $\tau^{ij}$  is precisely the first term on the right hand side of Eq. (5):

$$-\tau^{ij}{}_{,j} = \frac{1}{2} (F_{-}^{ij} + F_{+}^{ij}) \varrho u_j. \quad (7)$$

DIRAC identified the second term on the right hand side of Eq. (5) as being the term responsible for radiation effects in the limiting case of a point particle and in<sup>1</sup> it is shown that for an arbitrary body,

<sup>1</sup> E. ARNOUS, J. MADORE, and A. PAPAPETROU, Nuovo Cimento [1967] (To appear).

<sup>2</sup> P. A. M. DIRAC, Proc. Roy. Soc. London A 167, 148 [1938].



the tensor  $\tau^{ij}$  is nonradiative provided the duration of accelerated motion is finite. Therefore to the extent that we are not interested in studying the details of the electromagnetic forces in the interior and in the neighborhood of the body but rather those forces which are directly responsible for the movement of the body as a whole and for radiation, we are justified in using the identity (7) to write Eq. (5) in the form

$$(T^{ij} + \tau^{ij})_{,j} = \frac{1}{2} (F_-^{ij} - F_+^{ij}) \varrho u_j, \quad (8)$$

and in considering  $T^{ij} + \tau^{ij}$  as the total or renormalized matter tensor of the body.

Let  $x_0$  be a point in the interior of the body and  $\sigma$  an arbitrary spacelike hyperplane containing  $x_0$ . We form the renormalized energy-momentum vector and angular-momentum tensor the usual way but using  $T^{ij} + \tau^{ij}$  instead of  $T^{ij}$ :

$$P^i(\sigma, x_0) = \int_{\sigma} (T^{ij} + \tau^{ij}) n_j d\sigma, \quad (9)$$

$$M^{ij}(\sigma, x_0) = \int_{\sigma} X^{li} (T^{jl} + \tau^{jl}) n_k d\sigma, \quad (10)$$

where  $X^i$  is the vector joining the point of integration to  $x_0$  and  $n^i$  the unit normal to  $\sigma$ .

These equations may be written in the form

$$P^i(\sigma, x_0) = P_{(0)}^i(\sigma, x_0) + \delta P^i(\sigma, x_0), \quad (11)$$

$$M^{ij}(\sigma, x_0) = M_{(0)}^{ij}(\sigma, x_0) + \delta M^{ij}(\sigma, x_0), \quad (12)$$

where  $P_{(0)}^i$  and  $M_{(0)}^{ij}$  are the usual energy-momentum vector and angular-momentum tensor. We are interested here in finding the form of the correction terms  $\delta P^i$  and  $\delta M^{ij}$  in two particular simple limiting cases.

## II.

For each point  $x_0$  in the world tube of the body we have introduced a family of vectors  $P^i(\sigma, x_0)$  depending on the choice of hyperplane  $\sigma$ . We shall now choose from this family, one vector which we shall take to be the energy-momentum vector at the point  $x_0$ . (See for example <sup>3</sup> for a more detailed discussion in the case of General Relativity and <sup>4</sup> for a discussion of the case of a point particle.)

Let  $n^i$  be a timelike vector and  $\sigma_n$  be the hyperplane whose unit normal is parallel to  $n^i$ . Since we are considering macroscopic bodies, physical considerations require that  $P^i(\sigma_n, x_0)$  be timelike. There-

fore the map

$$n^i \rightarrow P^i(\sigma_n, x_0) / |P(\sigma_n, x_0)|$$

is a map of the forward light cone into itself. It can be shown using BROUWER's Fixed Point theorem that this map has a fixed point. We shall assume that it is unique and designate simply as  $P^i(x_0)$  the energy-momentum vector so formed. By construction therefore  $P^i = |P| n^i$ . We designate as  $M^{ij}(x_0)$  the angular-momentum vector formed using this hyperplane.

The two cases we are going to consider are the point particle, which we shall take as the limit of a body whose diameter tends to zero and the line segment rotating about a fixed point, which we shall take as the limit of a bar whose cross-sectional width tends to zero. Since these two cases involve extended bodies it is convenient to have a timelike curve  $C$  in the interior of the world tube to represent the motion. We construct this curve  $C$  as follows.

It is easily seen that the vector field  $M^{ij} P_j$  which is defined everywhere in the world tube of the body, is a spacelike field. Using the Fixed Point Theorem previously mentioned and the fact that the matter density is nowhere negative, one shows that it has a zero on every spacelike hyperplane. One can also show that this zero is unique. We designate by  $C$  the locus of the zeros. One sees immediately that  $C$  is a differentiable timelike curve which generalizes the center-of-mass line of NEWTONIAN Physics. Let  $t$  be the geodesic parameter along  $C$  and  $a^i$  the unit tangent vector. We shall assume for simplicity that  $a^i$  and  $P^i$  are parallel.

## III.

Let  $x$  and  $x'$  be two arbitrary points of a spacelike hyperplane  $\sigma$  in the interior of the world tube of the body. Let  $x_{\pm}$  be the points respectively on the future and past light cones with vertex  $x'$  which lie on the world line representing the motion of the matter through  $x$ .

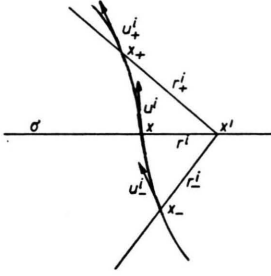
Let  $u^i$  be the particle velocity vector at  $x$  and define the vectors  $u_{\pm}^i$  by

$$u_{\pm}^i(x, x') = u^i(x_{\pm}).$$

$$\text{Set } r^i = x^{i'} - x^i \quad \text{and} \quad r_{\pm}^i = x^{i'} - x_{\pm}^i.$$

<sup>3</sup> J. MADORE, C. R. Acad. Sci. Paris **263**, 746 [1966].

<sup>4</sup> F. ROHRLLICH, Classical Charged Particles, Addison-Wesley, London 1965.



The advanced and retarded solutions of the wave equation,  $\varphi_{\pm}^i$  may be written in the form

$$\varphi_{\pm}^i = \mp \int_{\sigma} \frac{\varrho u_{\pm}^i}{u_{\pm} \cdot r_{\pm}} u \cdot n d\sigma. \quad (13)$$

A straightforward calculation shows that the advanced and retarded MAXWELL fields are given by

$$F_{\pm}^{ij} = \mp \int_{\sigma} \frac{\varrho u \cdot n}{r_{\pm} \cdot u_{\pm}} \left( \dot{u}_{\pm}^{[j} r_{\pm}^{i]} + \frac{1 - r_{\pm} \cdot \dot{u}_{\pm}}{r_{\pm} \cdot u_{\pm}} u_{\pm}^{[j} r_{\pm}^{i]} \right) d\sigma, \quad (14)$$

where  $\dot{u}^i = u^i_{,j} u^j$  is the acceleration vector. These may be written in the form

$$F_{\pm}^{ij} = \mp \int_{\sigma} \varrho \omega_{\pm}^{ij} u \cdot n d\sigma, \quad (15)$$

where  $\omega_{\pm}^{ij}$  are defined by

$$\omega_{\pm}^{ij} = \frac{1}{r_{\pm} \cdot u_{\pm}} \left( \dot{u}_{\pm}^{[j} r_{\pm}^{i]} + \frac{1 - r_{\pm} \cdot \dot{u}_{\pm}}{r_{\pm} \cdot u_{\pm}} u_{\pm}^{[j} r_{\pm}^{i]} \right). \quad (16)$$

We wish to calculate the correction terms

$$\delta P^i = \int_{\sigma} \tau^{ij} n_j d\sigma, \quad (17)$$

$$\text{and} \quad \delta M^{ij} = \int_{\sigma} X^{[i} \tau^{j]k} n_k d\sigma, \quad (18)$$

along the curve C. It is more convenient to calculate the right hand sides of (17) and (18) if we first differentiate both sides with respect to  $t$  and use Eq. (7) to eliminate  $\tau^{ij}$ . In the case when C is not a straight line, care must be taken in performing this derivation since the orientation of the hyperplane  $\sigma$  will depend on  $t$ .

By definition

$$\frac{d(\delta P^i)}{dt} = \lim_{t' \rightarrow t} \frac{\delta P^i(t') - \delta P^i(t)}{t' - t}.$$

This gives, using GAUSS' Theorem on the right hand side of (17),

$$\frac{d(\delta P^i)}{dt} = \lim_{t' \rightarrow t} \int_V \tau^{ij}_{,j} dv, \quad (19)$$

where  $V$  is the volume enclosed by the two hyperplanes  $\sigma(t')$  and  $\sigma(t)$ . We have here used the fact that  $\tau^{ij}$  vanishes as  $1/R^4$  at infinity along spacelike hyperplanes. A straightforward calculation shows that

$$dv = (t' - t) (1 - X \cdot \dot{a}) d\sigma + o((t' - t)^2) \quad (20)$$

where  $\dot{a}^i = a^i_{,j} a^j$ . Therefore we have

$$\frac{d(\delta P^i)}{dt} = \int_{\sigma} \tau^{ij}_{,j} (1 - X \cdot \dot{a}) d\sigma. \quad (21)$$

Using (7) this gives

$$\frac{d(\delta P^i)}{dt} = - \frac{1}{2} \int_{\sigma} \varrho (F_{-}^{ij} + F_{+}^{ij}) u_j (1 - X \cdot \dot{a}) d\sigma. \quad (22)$$

Similarly for the correction term  $\delta M^{ij}$  we find the following derivative along C:

$$\begin{aligned} \frac{d(\delta M^{ij})}{dt} = & \\ & - \frac{1}{2} \int_{\sigma} \varrho X^{[i} (F_{-}^{j]k} + F_{+}^{j]k}) u_k (1 - X \cdot \dot{a}) d\sigma. \end{aligned} \quad (23)$$

We have here used the assumption of section II that  $P^i$  and  $a^i$  are parallel.

Using (15) and defining  $\varphi^{ij}$  by

$$\varphi^{ij} = \frac{1}{2} (\omega_{-}^{ij} - \omega_{+}^{ij}), \quad (24)$$

we obtain the following expression for the derivatives of the two correction terms along C:

$$\frac{d(\delta P^i)}{dt} = - \iint \varrho \varrho' \varphi^{ij} u'_j (1 - X' \cdot \dot{a}) u \cdot a d\sigma d\sigma', \quad (25)$$

$$\begin{aligned} \frac{d(\delta M^{ij})}{dt} = & - \iint \varrho \varrho' X'^{[i} \varphi^{j]k} u'_k \\ & \cdot (1 - X' \cdot \dot{a}) u \cdot a d\sigma d\sigma'. \end{aligned} \quad (26)$$

The functions  $\varphi^{ij}$  are expressed in terms of the vectors  $u_{\pm}^i, \dot{u}_{\pm}^i, r_{\pm}^i$  defined at  $x_{\pm}$  on the future and past light cones with vertex  $x'$ . We wish to express them in terms of  $u^i, \dot{u}^i$  and  $r^i$ , vectors defined at  $x$  on the hyperplane  $\sigma$ . To do this we must expand the advanced and retarded quantities in a TAYLOR series about the point  $x$ . This calculation is straightforward but tedious and the details are given in the appendix.

The result is:

$$\begin{aligned} \varphi^{ij} = & \frac{1 - r \cdot \dot{u}}{R^3} (1 - \frac{1}{8} R^2 \dot{u}^2) r^{[i} u^{j]} + \frac{u^{[i} \dot{u}^{j]} - r^{[i} \ddot{u}^{j]}}{2R(1 - r \cdot \dot{u})} \\ & + (r \cdot u / R^3) (r^{[i} \dot{u}^{j]} - \frac{1}{2} r \cdot u u^{[i} \dot{u}^{j]}) \\ & + (1/R) o(r \cdot u) + o(|r|), \end{aligned} \quad (27)$$

where  $R$  is given by  $R = \sqrt{r \cdot u^2 - (1 - r \cdot \ddot{u}) r^2}$  and by  $o(u \cdot r)$  we designate a function which vanishes with  $u \cdot r$ .

#### IV.

We are now in a position to carry out the calculation of the correction term  $\delta P^i$  and  $\delta M^{ij}$  in the two limiting cases to be considered.

First we consider the case of a point particle. We calculate only the correction term  $\delta P^i$  in this case. We consider the point particle as the limit of an arbitrary body of diameter  $d$  with a charge distribution symmetric about the center of mass point  $x_0$  and which we assume to be in translational movement. That is we assume that the velocity vector field  $u^i$  is constant on  $\sigma$  and equal to  $a^i$ .

Therefore since  $r^i$  lies in  $\sigma$  and  $a^i$  is normal to  $\sigma$  we have  $u \cdot r = 0$  and it is easily seen from formula (27) that the quantity  $\varphi^{ij} u_j' u \cdot a$  appearing in the integrand of Eq. (25) is given by

$$\varphi^{ij} u_j' u \cdot a = (1 - r \cdot \dot{a}) (R^{-3} + \frac{3}{8} \dot{a}^2 / R) r^i - \dot{a}^i / [2 R (1 - r \cdot \dot{a})] + o(|r|). \quad (28)$$

Since  $r \cdot u = 0$  we have  $R = |r| \sqrt{1 - r \cdot \ddot{a}}$ . The vector  $X^i = x^i - x_0^i$  may be written as

$$X^i = \frac{1}{2} r^i + t^i$$

where  $t^i = \frac{1}{2}(x'^i + x^i) - x_0^i$  is a symmetric function of the two variables of integration. Therefore one deduces immediately the following expression for the integrand of Eq. (25):

$$\varrho \varrho' \varphi^{ij} u_j' (1 - X' \cdot \dot{a}) u \cdot a = \varrho \varrho' \left( -\frac{\dot{a}^i}{2|r|} + \frac{3 \dot{a}^2 r^i}{8|r|} - \frac{r \cdot \dot{a} \dot{a}^i}{2|r|} + \frac{t \cdot \dot{a}}{2|r|} \left( \dot{a}^i - \frac{r \cdot \dot{a} r^i}{|r|^2} \right) \right) + o(|r|). \quad (29)$$

Since  $r^i$  is antisymmetric in the two variables of integration the second and third terms on the right hand side contribute nothing to the integration. From the assumed symmetry of the charge density about the point  $x_0$  one sees immediately that the fourth term contributes also nothing. Therefore the expression (25) for  $d(\delta P^i)/dt$  becomes

$$\frac{d(\delta P^i)}{dt} = \frac{1}{2} \left( \iint \frac{\varrho \varrho'}{|r|} d\sigma d\sigma' \right) \dot{a}^i + o(d), \quad (30)$$

where  $o(d)$  is a quantity which vanishes with  $d$ .

The coefficient of  $\dot{a}^i$  is the electrostatic self-energy of the body,  $E_{(e)}$ . We have therefore

$$\frac{d(\delta P^i)}{dt} = E_{(e)} \dot{a}^i + o(d). \quad (31)$$

Since  $P^i$  and  $a^i$  have been assumed to be parallel, Eq. (11) may in this case be written in the form

$$m a^i = (m_{(0)} + E_{(e)}) a^i + o(d), \quad (32)$$

where  $m_{(0)} = |P_{(0)}|$  and  $m = m_{(0)} + E_{(e)}$  is the renormalized mass given by DIRAC.

#### V.

The second case to be considered is that of a charged line segment rotating about a fixed point of symmetry. Here we calculate only the correction term  $\delta M^{ij}$ . In fact we calculate only the space-like components  $\delta M^{a\beta}$  of this tensor; the components  $\delta M^{0\alpha}$  vanish by symmetry. The line  $C$  will be in

this case the world-line of the point of symmetry. We choose for convenience a system of coordinates such that it is the line  $x^\alpha = 0$  ( $\alpha = 1, 2, 3$ ).

We consider the line segment as the limit of a bar with a point symmetry whose cross-sectional width  $d$  tends to zero. The situation is more complicated in this case and we shall only be able to calculate  $\delta M^{a\beta}$  to within terms which remain bounded as  $d$  tends to zero and then only under rather restrictive assumptions on the asymptotic behaviour of the charge distribution.

We see from formula (27) that the quantity  $\varphi^{jk} u_k'$  appearing in the integrand of Eq. (26) is given by

$$\varphi^{jk} u_k' = ((1 - r \cdot \ddot{u})/R^3) (r^j u \cdot u' - u^j r \cdot u') - \ddot{u}^j r \cdot u^2 / 2 R^3 - \ddot{u}^j / 2 R + o(1), \quad (33)$$

where  $o(1)$  is a quantity which remains bounded in the limit as  $|r| \rightarrow 0$ . We have used the fact that  $u' \cdot \ddot{u} = o(|r|)$  and that  $u' \cdot u = 1 + o(|r|)$ .

In the limit  $d = 0$  the quantity  $r \cdot u$  is zero since  $r^i$  is a vector in the line segment and  $u^i$  is perpendicular to it. However, since this quantity appears as a factor in integrals which diverge in general as  $d \rightarrow 0$ , the asymptotic behaviour of the integral of the terms involving  $r \cdot u$  on the right hand side of Eq. (34) is obscure. It will in fact depend heavily on the asymptotic behaviour of the charge density. We shall not enter into a discussion of this point but shall assume the asymptotic behaviour of  $\varrho$  which is most con-

venient. One sees immediately that, since we have the equality  $r \cdot u = 0$  when both variables of integration lie on the plane defined by the axis of rotation and the axis of the rod, if we assume the charge to become concentrated in the limit on this plane we may set  $r \cdot u = 0$  under the integration sign.

Therefore neglecting relativistic corrections and taking the three dimensional dual we obtain for Eq. (26) the following

$$\begin{aligned} d(\delta \mathbf{M})/dt = & \frac{1}{2} \iint (\rho \rho' / |r|) \mathbf{X} \times \dot{\mathbf{u}} d\sigma d\sigma' \\ & - \frac{1}{2} \iint (\rho \rho' / |r|^3) \mathbf{X}' \times \mathbf{X} r \cdot \dot{\mathbf{u}} d\sigma d\sigma' \quad (35) \\ & + \iint (\rho \rho' / |r|^3) \mathbf{X}' \times \mathbf{X} d\sigma d\sigma' + o(1). \end{aligned}$$

By  $o(1)$  we designate here a quantity which remains bounded in the limit  $d \rightarrow 0$ .

Since  $\mathbf{X}' \times \mathbf{X}$  is antisymmetric in the two variables of integration one sees that the last term on the right hand side vanishes.

Also from our assumptions on the asymptotic behaviour of the charge distribution the second integral can give only a contribution perpendicular to the rotation vector and therefore must vanish because of the point symmetry of the bar. We are therefore left with the first term. Since the rotation vector  $\boldsymbol{\omega}$  and its derivative  $\dot{\boldsymbol{\omega}}$  are parallel, we have  $\mathbf{X} \times \dot{\mathbf{u}} = \mathbf{X}^2 \dot{\boldsymbol{\omega}}$ . Therefore Eq. (35) becomes

$$d(\delta \mathbf{M})/dt = \frac{1}{2} (\iint (\rho \rho' / |r|) \mathbf{X}^2 d\sigma d\sigma') \dot{\boldsymbol{\omega}} + o(1). \quad (36)$$

The coefficient of  $\dot{\boldsymbol{\omega}}$  is the moment of inertia  $I_{(e)}$  of the electrostatic energy about the origin. We have therefore

$$d(\delta \mathbf{M})/dt = I_{(e)} \dot{\boldsymbol{\omega}} + o(1). \quad (37)$$

If  $I_{(e)}$  is constant along  $C$ , (37) may be integrated to give the following expression for the dual of the space-like components of the correction term  $\delta M^{ij}$ :

$$\delta \mathbf{M} = I_{(e)} \boldsymbol{\omega} + o(1). \quad (38)$$

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## Appendix

We give here a derivation of formula (27). Let  $s_{\pm}$  be the proper time along the particle line between  $x$  and  $x_{\pm}$ .

We have the following expansions in terms of the proper time interval  $s_{\pm}$ :

$$r_{\pm}^i = r^i - s_{\pm} u^i - \frac{1}{2} s_{\pm}^2 \ddot{u}^i - \frac{1}{6} s_{\pm}^3 \dddot{u}^i + o(s_{\pm}^4), \quad (1A)$$

$$u_{\pm}^i = u^i + s_{\pm} \dot{u}^i + \frac{1}{2} s_{\pm}^2 \ddot{u}^i + o(s_{\pm}^3), \quad (2A)$$

$$\ddot{u}_{\pm}^i = \ddot{u}^i + s_{\pm} \dddot{u}^i + o(s_{\pm}^2). \quad (3A)$$

Using these we obtain the following expansions for  $r_{\pm}^{[i} \dot{u}_{\pm}^{j]}$  and for  $r_{\pm}^{[i} u_{\pm}^{j]}$ :

$$r_{\pm}^{[i} \dot{u}_{\pm}^{j]} = r^{[i} \dot{u}^{j]} - s_{\pm} u^{[i} \ddot{u}^{j]} + s_{\pm} r^{[i} \ddot{u}^{j]} - s_{\pm}^2 u^{[i} \ddot{u}^{j]} + o(s_{\pm}^3), \quad (4A)$$

$$r_{\pm}^{[i} u_{\pm}^{j]} = r^{[i} u^{j]} + s_{\pm} r^{[i} \dot{u}^{j]} - \frac{1}{2} s_{\pm}^2 u^{[i} \ddot{u}^{j]} + \frac{1}{2} s_{\pm}^2 r^{[i} \ddot{u}^{j]} - \frac{1}{6} s_{\pm}^3 u^{[i} \ddot{u}^{j]} + o(s_{\pm}^4). \quad (5A)$$

Equations (4A) and (5A) may be used in turn to give a partial expansion of the right hand side of Eq. (16) which defines the quantities  $\omega_{\pm}^{ij}$ :

$$\begin{aligned} \omega_{\pm}^{ij} = & A_{\pm} r^{[i} u^{j]} + \left( \frac{1}{r_{\pm} \cdot u_{\pm}^2} + s_{\pm} A_{\pm} \right) r^{[i} \dot{u}^{j]} - \left( \frac{s_{\pm}}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^2 A_{\pm}}{2} \right) u^{[i} \dot{u}^{j]} \\ & + \left( \frac{s_{\pm}}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^2 A_{\pm}}{2} \right) r^{[i} \ddot{u}^{j]} - \left( \frac{s_{\pm}^2}{r_{\pm} \cdot u_{\pm}^2} + \frac{s_{\pm}^3 A_{\pm}}{3} \right) u^{[i} \ddot{u}^{j]} + o(s_{\pm}). \end{aligned} \quad (6A)$$

We have set

$$(1 - r_{\pm} \cdot \dot{u}_{\pm}) / r_{\pm} \cdot u_{\pm}^3 = A_{\pm}.$$

To calculate the coefficients in this expansion, we need the following two expressions which are obtained from (1A), (2A) and (3A):

$$1 - r_{\pm} \cdot \dot{u}_{\pm} = 1 - r \cdot \dot{u} - s_{\pm} r \cdot \ddot{u} - \frac{1}{2} s_{\pm}^2 \ddot{u}^2 + o(s_{\pm}^3), \quad (7A)$$

$$r_{\pm} \cdot u_{\pm} = r \cdot u - s_{\pm} (1 - r \cdot \dot{u}) + \frac{1}{2} s_{\pm}^2 r \cdot \ddot{u} + \frac{1}{6} s_{\pm}^3 \ddot{u}^2 + o(s_{\pm}^4). \quad (8A)$$

The proper time intervals  $s_{\pm}$  are obtained as functions of  $r^i$  by using the fact that the vectors  $r_{\pm}^i$  are null. That is that

$$(r^i - s u^i - \frac{1}{2} s^2 \ddot{u}^i - \frac{1}{6} s^3 \ddot{u}^i)^2 = o(s^4). \quad (9A)$$

The solutions to this equation are

$$s_{\pm} = ((r \cdot u \pm R)/(1 - r \cdot \dot{u})) (1 \pm \frac{1}{6} R r \cdot \ddot{u} + \frac{1}{24} R^2 \ddot{u}^2) + R^2 o(r \cdot u) + o(|r|^4), \quad (10 A)$$

where we have set  $R = \sqrt{r \cdot u^2 - (1 - r \cdot \dot{u})^2}$  and where by  $o(u \cdot r)$  we designate a function which vanishes with  $u \cdot r$ .

Putting this expression for  $s_{\pm}$  in (8 A) we obtain the following expansion for  $r_{\pm} \cdot u_{\pm}$ :

$$r_{\pm} \cdot u_{\pm} = \mp R + \frac{1}{3} R^2 r \cdot \ddot{u} \pm \frac{1}{8} R^3 \ddot{u}^2 + o(|r|^4) + R^2 o(r \cdot u). \quad (11 A)$$

This and a similar expansion for  $1 - r_{\pm} \cdot \dot{u}_{\pm}$  give the following expansion for  $A_{\pm}$ :

$$A_{\pm} = ((1 - r \cdot \dot{u})/(\mp R^3)) (1 - \frac{1}{8} R^2 \ddot{u}^2) + o(|r|^0) + (1/R) o(r \cdot u). \quad (12 A)$$

The expansions of the retarded and advanced coefficients appearing in (6 A) are, using (10 A), (11 A) and (12 A):

$$1/r_{\pm} \cdot u_{\pm}^2 + s_{\pm} A_{\pm} = \mp r \cdot u/R^3 + (1/R) o(r \cdot u) + o(|r|^0), \quad (13 A)$$

$$s_{\pm}/r_{\pm} \cdot u_{\pm}^2 + \frac{1}{2} s_{\pm}^2 A_{\pm} = \pm 1/(2 R(1 - r \cdot \dot{u})) + (1/R) o(r \cdot u) + o(|r|), \quad (14 A)$$

$$s_{\pm}^2/r_{\pm} \cdot u_{\pm}^2 + \frac{1}{3} s_{\pm}^3 A_{\pm} = \frac{2}{3} + (1/R) o(r \cdot u) + o(|r|). \quad (15 A)$$

Putting these expressions into (6 A) we obtain finally the following series expansion for the quantities  $\omega_{\pm}^{ij}$ :

$$\begin{aligned} \omega_{\pm}^{ij} = & ((1 - r \cdot \dot{u})/(\mp R^3)) (1 - \frac{1}{8} R^2 \ddot{u}^2) r^{[i} u^{j]} \mp 1/(2 R(1 - r \cdot \dot{u})) (u^{[i} \ddot{u}^{j]} - r^{[i} \ddot{u}^{j]}) \\ & - \frac{2}{3} u^{[i} \ddot{u}^{j]} \mp (r \cdot u/R^3) (r^{[i} \dot{u}^{j]} - \frac{1}{2} r \cdot u u^{[i} \dot{u}^{j]}) + (1/R) o(r \cdot u) + o(|r|). \end{aligned} \quad (16 A)$$

Using this and the definition of  $\varphi^{ij}$  (24) we obtain immediately the desired expansion:

$$\begin{aligned} \varphi^{ij} = & ((1 - r \cdot \dot{u})/R^3) (1 - \frac{1}{8} R^2 \ddot{u}^2) r^{[i} u^{j]} + 1/(2 R(1 - r \cdot \dot{u})) (u^{[i} \dot{u}^{j]} - r^{[i} \ddot{u}^{j]}) \\ & + (r \cdot u/R^3) (r^{[i} \dot{u}^{j]} - \frac{1}{2} r \cdot u u^{[i} \dot{u}^{j]}) + (1/R) o(r \cdot u) + o(|r|). \end{aligned} \quad (27)$$